# BRUNN-MINKOWSKI AND ZHANG INEQUALITIES FOR CONVOLUTION BODIES

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Abstract. A quantitative version of Minkowski sum, extending the definition of $\theta$-convolution of convex bodies, is studied to obtain extensions of the Brunn-Minkowski and Zhang inequalities, as well as, other interesting properties on Convex Geometry involving convolution bodies or polar projection bodies. The extension of this new version to more than two sets is also given.

Zhang inequality, Brunn-Minkowski inequality, Convolution Body.

## 1. Introduction and motivation

The Minkowski sum of two sets $K, L \subseteq \mathbb{R}^n$ is defined as the set

$$K + L = \{x \in \mathbb{R}^n \,:\, K \cap (x - L) \neq \emptyset\}.$$

The essential sum in terms of measure is defined as

$$K +_e L = \{x \in \mathbb{R}^n : |K \cap (x - L)| > 0\},$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$. A quantitative version of this definition, involving the proportional measure of the intersections, gives the following subset of $K + L$

$$K +_\theta L = \{x \in K + L : |K \cap (x - L)| \geq \theta M(K, L)\}$$

for $\theta \in [0, 1]$, where $M(K, L) := \max_{x \in K+L} |K \cap (x - L)|$. This set is called the *$\theta$-convolution set* of $K$ and $L$. Note that $K +_0 L$ is the usual Minkowski sum $K + L$. When $K$ and $L$ are symmetric convex bodies, this definition coincides with the definition of convolution bodies used in [10, 18, 13, 19, 22]. However, our notation differs from the one used there, in order to emphasize the connection with the standard Minkowski sum. Properties of $\theta$-convolution bodies are given in Section 2.

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Our purpose is to find volume estimates, from above and below, of the $\theta$-convolution of two sets. In what follows we will motivate our interest in studying the volume of this family of sets.

The celebrated Brunn-Minkowski inequality

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

for two non empty measurable sets $A, B \subset \mathbb{R}^n$ has been widely applied to solve a large number of problems involving geometrical quantities such as volume, surface area, and mean width. In the last thirty years the Brunn-Minkowski inequality has become an essential analytical tool to develop the so-called Local Theory of Normed Spaces and Convex Geometric Analysis [9, 20, 2, 14].

In Section 3 a generalization of the Brunn-Minkowski inequality is studied. Even though extensive work with this inequality as backbone have emerged both within the class of convex bodies [7, 11, 12] and under other settings [3, 24, 23, 4], we pursuit something closer in spirit to [1, 21]. See [8] for a comprehensive survey on the Brunn-Minkowski inequality including extensions, applications and its relation to other analytical inequalities.
Namely, we pose the problem of finding the best function $\varphi_n(\theta)$ such that

$$|A +_\theta B|^{1/n} \geq \varphi_n(\theta)^{1/n}(|A|^{1/n} + |B|^{1/n}) \tag{1.1}$$

for any non empty measurable sets $A, B \subset \mathbb{R}^n$. It is proved that $\varphi_n(\theta) = (1-\theta^{1/n})^n$ satisfies (1.1) when $A, B$ are convex bodies. Some particular cases are also studied.

Following the work of Kiener [10], Schmuckenschläger [18] proved that for any convex body $K$ of volume 1,

$$(1-\theta)\Pi^*(K) \subseteq K +_\theta (-K) \subseteq \log\frac{1}{\theta}\Pi^*(K) \tag{1.2}$$

where $\Pi^*(K)$ is the polar projection body of $K$, the unit ball of the norm $\|x\|_{\Pi^*(K)} = |x||P_{x^\perp}K|$. Here $P_{x^\perp}$ denotes the orthogonal projection on the hyperplane orthogonal to $x$.

These inclusions imply $|K|\Pi^*(K) = \lim_{\theta\to 1^-} \dfrac{K +_\theta (-K)}{1-\theta}$ in the Haussdorff metric for any convex body $K$.

In Section 4 we modify the argument to improve the estimate (1.2) (see Proposition 4.1)

$$(1-\theta)|K|\Pi^*(K) \subseteq K +_\theta (-K) \subseteq n(1-\theta^{\frac{1}{n}})|K|\Pi^*(K).$$

The most famous inequality concerning the volume of the polar projection body of a convex body $K \subset \mathbb{R}^n$ is Petty projection inequality,

$$|K|^{n-1}|\Pi^*(K)| \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n$$

where $\omega_n$ denotes the volume of the $n$-dimensional Euclidean ball. The equality is attained provided $K$ is an ellipsoid (See [15]). A different proof using convolutions can be found in [19].

In [25], Zhang proved a reverse form of this inequality

$$|K|^{n-1}|\Pi^*(K)| \geq \frac{1}{n^n}\binom{2n}{n} \tag{1.3}$$

for any convex body, with equality if and only if $K$ is a simplex. Zhang inequality can be written as

$$\left|\lim_{\theta\to 1^-} \frac{K +_\theta (-K)}{1-\theta}\right| \geq \frac{1}{n^n}\binom{2n}{n}|K|. \tag{1.4}$$

In [22], Tsolomitis studied the existence of limiting convolution bodies

$$\lim_{\theta\to 1^-} \frac{K +_\theta L}{(1-\theta)^\alpha} \tag{1.5}$$

for symmetric convex bodies $K$ and $L$, and some exponent $\alpha$. He proved that under some regularity assumptions, there exists the limit (1.5) with $\alpha = 1$, denoted by $C(K, L)$.

In [16], Rogers and Shephard obtained the inequality

$$|K - K| \leq \binom{2n}{n}|K| \tag{1.6}$$

for any convex body $K$, with equality if and only if $K$ is a simplex. Throughout the proof it is showed that

$$K +_\theta (-K) \supseteq (1 - \theta^{1/n})(K - K) \tag{1.7}$$

with equality if and only if $K$ is a simplex. They also showed in [17] the extension for two different convex bodies

$$|K - L||K \cap L| \leq \binom{2n}{n}|K||L|. \tag{1.8}$$

The last part of Section 4 is devoted to generalize the inclusions stated in (1.2) and Zhang inequality (1.3) for limiting convolutions of different convex bodies, when they exist. This generalization is a consequence of Corollary 2.4, from which (1.8) can be obtained (see Remark 4.7).

We now give two more arguments showing the interest in estimating the volume of $\theta$-convolution sets.

(a) A symmetric convex body $B$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $A \subseteq B$

$$A + A_\varepsilon^c \subseteq 2(1-\delta)B$$

where $A_\varepsilon = \{x \in \mathbb{R}^n : \|x-y\|_B \le \varepsilon\,, \text{ for all } y \in A\}$.

Using $\theta$-convolution sets, we may give a quantitative definition of uniform convexity. We call $B$ *almost uniformly convex* if there exists $\theta \in [0,1)$ such that for every $\varepsilon > 0$ there exists $\delta(\varepsilon, \theta) > 0$ satisfying

$$A +_\theta A_\varepsilon^c \subseteq 2(1-\delta)B$$

for every $A \subseteq B$. Now, a good lower estimate for the volume of the $\theta$-convolution of two sets, such as Brunn-Minkowski type inequality (1.1) in the equivalent geometric form

$$\left|\frac{K +_\theta L}{2}\right| \ge \varphi_n(\theta)|K|^{\frac{1}{2}}|L|^{\frac{1}{2}}$$

for some function $\varphi_n$ would imply a concentration of measure phenomenon. Indeed, for any $A \subseteq B$ with $|A| \ge \frac{1}{2}$ and $B$ an almost uniformly convex body of volume 1, the inequalties

$$2\varphi_n(\theta)(|A_\varepsilon^c|/2)^{1/2n} \le 2\varphi_n(\theta)(|A_\varepsilon^c||A|)^{1/2n} \le |A +_\theta A_\varepsilon^c|^{1/n} \le 2(1-\delta)$$

would give

$$|A_\varepsilon^c| \le 2\left(\frac{1-\delta}{\varphi_n(\theta)}\right)^{2n}.$$

This inequality exhibits a concentration of measure phenomenon provided that $1-\delta < \varphi_n(\theta)$.

(b) In [6], Dar conjectured that for any two convex bodies $K, L \subset \mathbb{R}^n$

$$M(K,-L)^{1/n} + \frac{|K|^{1/n}|L|^{1/n}}{M(K,-L)^{1/n}} \le |K+L|^{1/n}. \tag{1.9}$$

Connecting the latter with volumes of $\theta$-convolutions, the conjecture (1.9) for two convex bodies $K$, $L = -L$ is true if and only if

$$\left(\int_0^1 |K +_\theta L| d\theta\right)^{1/n} \le \left|\frac{K+L}{2}\right|^{1/n} + \sqrt{\left|\frac{K+L}{2}\right|^{2/n} - |K|^{1/n}|L|^{1/n}}.$$

Indeed, consider $f(x) = x + \frac{|K|^{1/n}|L|^{1/n}}{x} - |K+L|^{1/n}$, $x > 0$. Since $f(x) = f\left(\frac{|K|^{1/n}|L|^{1/n}}{x}\right)$ holds for every $x > 0$,

$$f(M(K,L)^{1/n}) \leq 0 \text{ if and only if } f\left(\frac{|K|^{1/n}|L|^{1/n}}{M(K,L)^{1/n}}\right) \leq 0.$$

From the symmetry of L

$$\frac{|K|^{1/n}|L|^{1/n}}{M(K,L)^{1/n}} = \left(\int_0^1 |K +_\theta L| d\theta\right)^{1/n}.$$

Now $f(x) \leq 0$ if and only if $p(x) = x^2 - |K+L|^{1/n}x + |K|^{1/n}|L|^{1/n} \leq 0$. Thus, (1.9) is true if and only if

$$\left(\int_0^1 |K +_\theta L| d\theta\right)^{\frac{1}{n}} = \frac{|K|^{\frac{1}{n}}|L|^{\frac{1}{n}}}{M(K,L)^{\frac{1}{n}}}$$

belongs to

$$\left[\left|\tfrac{K+L}{2}\right|^{\frac{1}{n}} - \sqrt{\left|\tfrac{K+L}{2}\right|^{\frac{2}{n}} - |K|^{\frac{1}{n}}|L|^{\frac{1}{n}}}, \left|\tfrac{K+L}{2}\right|^{\frac{1}{n}} + \sqrt{\left|\tfrac{K+L}{2}\right|^{\frac{2}{n}} - |K|^{\frac{1}{n}}|L|^{\frac{1}{n}}}\right].$$

Since $M(K,L)^{1/n} \leq \frac{1}{2}|K+L|^{1/n}$, under the assumption (1.9) it is always true that

$$\left(\int_0^1 |K +_\theta L| d\theta\right)^{1/n} \geq \left|\frac{K+L}{2}\right|^{1/n} - \sqrt{\left|\frac{K+L}{2}\right|^{2/n} - |K|^{1/n}|L|^{1/n}}.$$

Thus, (1.9) is true if and only if

$$\left(\int_0^1 |K +_\theta L| d\theta\right)^{1/n} \leq \left|\frac{K+L}{2}\right|^{1/n} + \sqrt{\left|\frac{K+L}{2}\right|^{2/n} - |K|^{1/n}|L|^{1/n}}$$

as we wanted.

Therefore, giving good volume estimates for $K +_\theta L$ would prove (or disprove) Dar's conjecture. Note that the conjecture is also equivalent to

$$M(K,L)^{1/n} \geq \left|\frac{K+L}{2}\right|^{1/n} - \sqrt{\left|\frac{K+L}{2}\right|^{2/n} - |K|^{1/n}|L|^{1/n}}.$$

## 2. Properties of the $\theta$-convolution of convex bodies

In this section we give some properties of the $\theta$-convolution of two convex bodies, from which the Brunn-Minkowski-type inequality for the $\theta$-convolution of convex bodies $|K +_\theta L|^{1/n} \geq (1-\theta^{1/n})\left(|K|^{1/n}+|L|^{1/n}\right)$ follows. However, this bound does not seem to be sharp.

We now list some basic properties of $M(K,L)$ and the $\theta$-convolution in the following

**Proposition 2.1.** *Let $K, L \subset \mathbb{R}^n$ be compact sets, $\lambda \geq 0$, $\theta \in (0,1]$, $x \in \mathbb{R}^n$ and $T \in GL_n(\mathbb{R}) = \{T : \mathbb{R}^n \to \mathbb{R}^n \ : \ T \text{ is linear}\}$*

- (a1) $M(K,L) = M(L,K)$
- (a2) $M(x+K,L) = M(K,L)$
- (a3) $M(\lambda K, \lambda L) = \lambda^n M(K,L)$
- (a4) $M(TK,TL) = |\det T| M(K,L)$
- (a5) *If $K = -L$ or $K, L$ are symmetric, then $M(K,L) = |K \cap (-L)|$.*
- (b1) $(\lambda K) +_\theta (\lambda L) = \lambda(K +_\theta L)$
- (b2) $K +_\theta L = L +_\theta K$
- (b3) $(x+K) +_\theta L = x + (K +_\theta L)$
- (b4) $TK +_\theta TL = T(K +_\theta L)$

A first question about this $\theta$-convolution is the following: If $K$ and $L$ are both convex, is their $\theta$-convolution convex as well? The affirmative answer is a consequence of the following result. In what follows, using $(b3)$ in Proposition 2.1 above, we will assume without loss of generality, that $M(K,L) = |K \cap (-L)|$.

**Proposition 2.2.** *Let $K, L \subset \mathbb{R}^n$ be convex bodies such that $M(K,L) = |K \cap (-L)|$. Then for every $\theta_1, \theta_2, \lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$ we have that*

$$\lambda_1(K +_{\theta_1} L) + \lambda_2(K +_{\theta_2} L) \subseteq K +_{\theta_0} L,$$

*where* $1-\theta_0^{1/n} = \lambda_1(1-\theta_1^{1/n}) + \lambda_2(1-\theta_2^{1/n})$.

*Proof.* Let $x_1 \in K +_{\theta_1} L$ and $x_2 \in K +_{\theta_2} L$. Using the convexity of $K$ and $L$, we have

$$K \cap (\lambda_1 x_1 + \lambda_2 x_2 - L) \supseteq (1-\lambda_1-\lambda_2)[K \cap (-L)] + \lambda_1[K \cap (x_1 - L)] + \lambda_2[K \cap (x_2 - L)].$$

Taking volumes, using the classical Brunn-Minkowski inequality and the fact that $x_i \in K +_{\theta_i} L$ we have

$$|K \cap (\lambda_1 x_1 + \lambda_2 x_2 - L)| \geq [1 - \lambda_1(1-\theta_1^{1/n}) + \lambda_2(1-\theta_2^{1/n})]^n M(K,L). \qquad \square$$

**Remark 2.3.** *Taking* $\theta_1 = \theta_2$ *and* $\lambda_2 = 1 - \lambda_1$ *we have that* $K +_\theta L$ *is convex if* $K$ *and* $L$ *are. Also, taking* $\theta_1 = \theta_2$ *leads us to the following*

**Corollary 2.4.** *Let* $K, L \subset \mathbb{R}^n$ *be convex bodies such that* $M(K, L) = |K \cap (-L)|$*. Then, for every* $0 \le \theta \le \theta_0 < 1$ *we have*

$$\frac{K +_\theta L}{1 - \theta^{\frac{1}{n}}} \subseteq \frac{K +_{\theta_0} L}{1 - \theta_0^{\frac{1}{n}}}.$$

*Proof.* Taking $\theta_1 = \theta_2 = \theta$ in the above proposition, for any $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 \le 1$

$$(\lambda_1 + \lambda_2)(K +_\theta L) \subseteq K +_{\theta_0} L,$$

with $1 - \theta_0^{1/n} = (\lambda_1 + \lambda_2)(1 - \theta^{\frac{1}{n}})$. Since $\lambda_1 + \lambda_2 = \dfrac{1 - \theta_0^{\frac{1}{n}}}{1 - \theta^{\frac{1}{n}}}$,

$$\frac{1 - \theta_0^{\frac{1}{n}}}{1 - \theta^{\frac{1}{n}}}(K +_\theta L) \subseteq K +_{\theta_0} L$$

whenever $\lambda_1 + \lambda_2 \le 1$, which means $0 \le \theta \le \theta_0 \le 1$. □

## 3. Brunn-Minkowski type inequality for $\theta$-convolution bodies

From the previous study on convolution of two sets, the following natural question arises: what kind of Brunn-Minkowski-type inequality for $\theta$-convolutions

$$|K +_\theta L|^{1/n} \ge \varphi_n(\theta)^{1/n}(|K|^{1/n} + |L|^{1/n}). \tag{3.1}$$

does it hold?

As in the classical case, the homogeneity allows one to formulate the inequality in different equivalent forms.

**Proposition 3.1.** *The following statements are all equivalent:*

(*i*) *For* $K, L$ *measurables in* $\mathbb{R}^n$
$$|K +_\theta L|^{1/n} \ge \varphi_n(\theta)^{1/n}(|K|^{1/n} + |L|^{1/n}).$$
(*ii*) *For* $K, L$ *measurables in* $\mathbb{R}^n$ *and* $0 < \lambda < 1$
$$|\lambda K +_\theta (1 - \lambda)L|^{1/n} \ge \varphi_n(\theta)^{1/n}(\lambda|K|^{1/n} + (1 - \lambda)|L|^{1/n}).$$
(*iii*) *For* $K, L$ *measurables in* $\mathbb{R}^n$ *and* $0 < \lambda < 1$
$$|\lambda K +_\theta (1 - \lambda)L| \ge \varphi_n(\theta)(|K|^\lambda \cdot |L|^{1-\lambda}).$$

(*iv*) *For* $K, L$ *measurables in* $\mathbb{R}^n$ *and* $0 < \lambda < 1$

$$|\lambda K +_\theta (1-\lambda)L| \geq \varphi_n(\theta) \min\{|K|, |L|\}.$$

(*v*) *For* $K, L$ *measurables in* $\mathbb{R}^n$ *such that* $|K| = |L| = 1$ *and* $0 < \lambda < 1$

$$|\lambda K +_\theta (1-\lambda)L| \geq \varphi_n(\theta).$$

*Proof.* $(i) \to (ii)$ and $(iii) \to (iv) \to (v)$ are immediate. The proof of $(ii) \to (iii)$ is obtained by taking logarithm and using its concavity.

Finally, apply $(v)$ with $\overline{K} = |K|^{-1/n}K$, $\overline{L} = |L|^{-1/n}L$ and $\lambda = \frac{s|K|^{1/n}}{s|K|^{1/n}+t|L|^{1/n}}$, and use the homogeneity of the convolution (Proposition 2.1 (b1)) to get $(i)$. □

A first inequality in this direction for convex bodies is obtained from the following consequence of Proposition 2.2:

**Theorem 3.2.** *Let* $K, L \subset \mathbb{R}^n$ *be convex bodies. Then*

$$\theta^{\frac{1}{n}}(K +_1 L) + (1-\theta^{\frac{1}{n}})(K + L) \subseteq K +_\theta L. \tag{3.2}$$

*Proof.* Take $\theta_1 = 1$, $\theta_2 \to 0$, $\lambda_1 = \theta^{\frac{1}{n}}$, and $\lambda_2 = (1-\theta^{\frac{1}{n}})$ in Proposition 2.2 to obtain the desired result. □

If $0 \in K +_1 L$ we extend (1.7) from Rogers and Shephard's work. The condition $0 \in K +_1 L$ is equivalent to $M(K, L) = |K \cap (-L)|$, and that is verified under the assumptions of Proposition 2.1 $(a5)$.

As an immediate corollary we have:

**Corollary 3.3.** *Let* $K,\ L \subset \mathbb{R}^n$ *be convex bodies. Then*

$$|K +_\theta L|^{1/n} \geq (1-\theta^{1/n})(|K|^{1/n} + |L|^{1/n}).$$

*Equivalently,* $\varphi_n(\theta) \geq (1-\theta^{1/n})^n$ *in (3.1).*

*Proof.* Taking volumes in (3.2)

$$|K +_\theta L|^{1/n} \geq (1-\theta^{1/n})|K + L|^{1/n} \tag{3.3}$$

and applying Brunn-Minkowski inequality

$$(1-\theta^{1/n})|K + L|^{1/n} \geq (1-\theta^{1/n})(|K|^{1/n} + |L|^{1/n}) \tag{3.4}$$

we obtain the desired result. □

This first Brunn-Minkowski-type inequality, however, does not seem to be sharp. Indeed, if $K = -L$ equality holds in (3.3) provided that $K$ is a simplex (see [16]), which does not give equality in (3.4) (see equality cases in Brunn-Minkowski inequality in [5]). See examples at the end of the section for details.

The following result improves the inclusion

$$(1-\theta^{\frac{1}{n}})(K+L) \subseteq K +_\theta L \tag{3.5}$$

providing a new set between them. A good estimate for the volume of this new set would lead to a better estimate for $|K +_\theta L|$.

**Theorem 3.4.** *Let $K, L \subset \mathbb{R}^n$ be convex bodies such that $M(K,L) = |K \cap (-L)|$. Then for all $\theta \in [0,1]$,*

$$\begin{aligned} K +_\theta L &\supseteq \left\{ a+b \ : \ a \in K,\ b \in L,\ \frac{|(1-||a||_K)K \cap (1-||b||_L)(-L)|}{|K \cap (-L)|} \geq \theta \right\} \\ &\supseteq (1-\theta^{1/n})(K+L). \end{aligned}$$

*Proof.* Let $x \in K+L$, then $x = a+b$ with $a \in K$ and $b \in L$. From the convexity of $K$

$$(1-||a||_K)K + ||a||_K \frac{a}{||a||_K} \subseteq K$$

Also, since $||-b||_{-L} = ||b||_L$ and $L$ is convex

$$(1-||b||_L)(-L) + ||b||_L \frac{-b}{||b||_L} + x \subseteq x - L$$

Since $x - b = a$, we have

$$(1-||a||_K)K + a \subseteq K \text{ and } (1-||b||_L)(-L) + a \subseteq x - L$$

Thus,

$$a + (1-||a||_K)K \cap (1-||b||_L)(-L) \subseteq K \cap (x-L)$$

then

$$|K \cap (x-L)| \geq |(1-||a||_K)K \cap (1-||b||_L)(-L)|.$$

Consequently,

$$K +_\theta L \supseteq \{a+b \ : \ a \in K,\ b \in L,\ \frac{|(1-||a||_K)K \cap (1-||b||_L)(-L)|}{|K \cap (-L)|} \geq \theta\}.$$

This set trivially contains the set

$$\{a+b \ : \ \inf\{(1-||a||_K)^n, (1-||b||_L)^n\} \geq \theta\} = (1-\theta^{1/n})(K+L). \qquad \square$$

In order to get a more accurate idea of how good the bound in Corollary 3.3 is, we estimate the quotient $\dfrac{|K +_\theta L|^{1/n}}{|K|^{1/n} + |L|^{1/n}}$ for some particular pairs of bodies.

**Examples:**

1) For $K,\ L$ cubes whose sides are parallel to the coordinate hyperplanes, it is not hard to see that the quotient is minimized when $K=L=[-1/2,1/2]^n$, and its value equals to
$$\left[1-\theta\sum_{k=0}^{n-1}\frac{(-\log\theta)^k}{k!}\right]^{1/n}.$$
2) For $K=L$ the unit Euclidean ball, the quotient equals to $R_n(\theta)$ given by the equality
$$2\omega_{n-1}\int_{R_n(\theta)}^1\left(1-s^2\right)^{\frac{n-1}{2}}ds=\theta\omega_n$$
where $\omega_n$ denotes the volume of the $n$-dimensional unit Euclidean ball.
3) As it was mentioned above, in [16] it was proved that, for $K=L$ the simplex, the quotient equals to
$$(1-\theta^{1/n})\frac{\binom{2n}{n}^{1/n}}{2}\sim 2(1-\theta^{1/n}).$$

Comparing these three cases, it seems that the minimum value for the quotient is attained in a different case depending on $\theta$. This fact makes difficult to find a family of bodies in which the minimum is attained.

## 4. A connection with projection bodies and Zhang inequality

This section is devoted to generalize the inclusions (1.2) and Zhang inequality for convolution of different convex bodies.

The following result generalizes the right hand side inclusion in (1.2). We extend the ideas used in [18]

**Proposition 4.1.** *Let $K,\ L\subset\mathbb{R}^n$ be convex bodies such that $M(K,L)=|K\cap(-L)|$. Then, for every $\theta\in(0,1)$*
$$K+_\theta L\subseteq\left\{x\in\mathbb{R}^n\,:\,|x|\left|\frac{d^+}{dt}\left|K\cap\left(t\frac{x}{|x|}-L\right)\right|_{t=0}\right|\leq n(1-\theta^{\frac1n})M(K,L)\right\}.$$

*Proof.* The concavity of the function $x\mapsto|K\cap(x-L)|^{\frac1n}$ implies
$$\begin{aligned}|K\cap(\lambda x-L)| &\geq \left((1-\lambda)M(K,L)^{\frac1n}+\lambda|K\cap(x-L)|^{\frac1n}\right)^n\\ &= M(K,L)\left[1+\lambda\left(\frac{|K\cap(x-L)|^{\frac1n}}{M(K,L)^{\frac1n}}-1\right)\right]^n\\ &\geq M(K,L)\left[1+\lambda n\left(\frac{|K\cap(x-L)|^{\frac1n}}{M(K,L)^{\frac1n}}-1\right)\right]\end{aligned}$$

for $\lambda \in [0,1]$ and $x \in K+L$. On the other hand,

$$\begin{aligned} |K\cap(\lambda x - L)| &= M(K,L) + \int_0^{\lambda|x|} \frac{d^+}{dt}\left|K\cap\left(t\frac{x}{|x|}-L\right)\right| dt \\ &\le M(K,L) + \lambda|x| \max_{t\in[0,\lambda|x|]} \frac{d^+}{dt}\left|K\cap\left(t\frac{x}{|x|}-L\right)\right| \\ &= M(K,L) + \lambda|x|\frac{d^+}{dt}\left|K\cap\left(t\frac{x}{|x|}-L\right)\right|_{t=0} \end{aligned}$$

again using the concavity of $x \mapsto |K\cap(x-L)|^{\frac{1}{n}}$. Consequently, for any $\lambda \in (0,1]$,

$$nM(K,L)\left(\frac{|K\cap(x-L)|^{\frac{1}{n}}}{M(K,L)^{\frac{1}{n}}} - 1\right) \le |x|\frac{d^+}{dt}\left|K\cap\left(t\frac{x}{|x|}-L\right)\right|_{t=0}.$$

Since the lateral derivative is non positive, we get the desired inclusion. $\square$

**Remark 4.2.**

1) *If $L=-K$, then the right-hand side set is exactly $n(1-\theta^{\frac{1}{n}})|K|\Pi^*K$ which improves the right hand side inclusion in (1.2).*
2) *In general, when the limiting convolution body with $\alpha=1$ $C(K,L)$ exists, this set is $n(1-\theta^{\frac{1}{n}})C(K,L)$. Under this assumption the previous result can be deduced from Corollary 2.4 letting $\theta_0 \to 1^-$ (See proof of Theorem 4.6).*
3) *This set is not necessarily bounded. Nevertheless, a general inclusion $K +_\theta L \subseteq n(1-\theta^{\frac{1}{n}})C$, for some body $C$ independent from $\theta$ can not be proved, since it was shown in [22] that the limiting body with $\alpha=1$ could be non compact.*

The left-hand side inclusion in (1.2) is generalized with the following:

**Proposition 4.3.** *Let $K,\ L \subset \mathbb{R}^n$ be convex bodies such that $M(K,L) = |K\cap(-L)|$. Then, for every $\theta\in(0,1)$*

$$K +_\theta L \supseteq (1-\theta)M(K,L)conv(\Pi^*K \cup \Pi^*L).$$

*Proof.* In [13] it is proved that, given two convex bodies $K$, $L$, and $u\in S^{n-1}$, the function $f(r) = |K\cap(ru+L)|$ verifies $f'(0) = |C_u^+(1,2)| - |C_u^-(2,1)|$, where

$$\begin{aligned} C_u^+(1,2) &= P_u(K\cap L)\cap\{\psi_K^+ > \psi_L^+ \ge \psi_K^- > \psi_L^-\} \\ C_u^+(2,1) &= P_u(K\cap L)\cap\{\psi_L^+ \ge \psi_K^+ > \psi_L^- \ge \psi_K^-\} \end{aligned}$$

with

$$\psi_K^+(y) = \max\{t \ : \ tu+y\in K\}$$

and

$$\psi_K^-(y) = \min\{t \ : \ tu+y\in K\}.$$

Thus $\frac{d}{d\vec{v}}|K\cap(x-L)|\geq -|P_v^+K\cap(x-L)|\geq -\min\{|P_v^+K|,|P_v^+L|\}$. Consequently,

$$|K\cap(x-L)|=$$
$$M(K,L)+\int_0^{|x|}\frac{d}{dt}\left|K\cap\left(t\frac{x}{|x|}-L\right)\right|\geq M(K,L)-|x|\min\{|P_v^+K|,|P_v^+L|\}.$$

Hence, if $M(K,L)-|x|\min\{|P_v^+K|,|P_v^+L|\}\geq\theta M(K,L)$ then $x\in K+_\theta L$ and this holds if and only if $\min\{||x||_{\Pi^*K},||x||_{\Pi^*L}\}\leq(1-\theta)M(K,L)$.

So, $(1-\theta)M(K,L)(\Pi^*K\cup\Pi^*L)\subseteq K+_\theta L$. The convexity of the set $K+_\theta L$ yields the desired result. □

**Remark 4.4.** *Taking $L=-K$ and $|K|=1$, we recover the left hand side inclusion in (1.2).*

**Remark 4.5.** *Applying Zhang inequality we deduce that if there exists the limiting convolution body $C(K,L)$, then*

$$\min\{|K|^{n-1},|L|^{n-1}\}\left|\frac{C(K,L)}{M(K,L)}\right|\geq\binom{2n}{n}\frac{1}{n^n}$$

*which extends Zhang inequality (1.4). Nevertheless, a stronger extension of Zhang inequality can be proved using Corollary 2.4.*

**Theorem 4.6.** *Let $K,L\subset\mathbb{R}^n$ be convex bodies such that $M(K,L)=|K\cap(-L)|$. If there exists $C(K,L)$ the limiting convolution body with exponent $\alpha=1$, then*

$$|C(K,L)|\geq\binom{2n}{n}\frac{1}{n^n}\frac{|K||L|}{M(K,L)}$$

*Proof.* From Corollary 2.4 we have that for every $0\leq\theta\leq\theta_0<1$

$$\frac{K+_\theta L}{1-\theta^{\frac{1}{n}}}\subseteq\frac{K+_{\theta_0}L}{1-\theta_0^{\frac{1}{n}}}.$$

Thus, letting $\theta_0\to1^-$ we obtain that for every $\theta\in[0,1)$

$$\frac{K+_\theta L}{1-\theta^{\frac{1}{n}}}\subseteq\lim_{\theta_0\to1^-}\frac{1-\theta_0}{1-\theta_0^{\frac{1}{n}}}\frac{K+_{\theta_0}L}{1-\theta_0}=nC(K,L),$$

and taking volumes

$$|K+_\theta L|\leq n^n(1-\theta^{\frac{1}{n}})^n|C(K,L)|$$

for $\theta\in[0,1)$. Integrating over $[0,1]$ yields the desired result. □

**Remark 4.7.** *Taking $\theta=0$ in Corollary 2.4, applying volumes and integrating in $\theta_0$ we obtain Rogers-Shephard inequality (1.8).*

## 5. Convolution of n bodies

In this section we will extend the definition of $\theta$-convolution bodies to more than two sets. The $\theta$-convolution is not associative (as a simple computation with Euclidean balls of different radius shows) so a definition of an $n$-fold convolution can not be made inductively. Nevertheless, since $|K \cap (x-L)| = \chi_K * \chi_L(x)$ and the convolution is associative, it seems natural to make the following extension of $\theta$-convolution bodies:

**Definition 5.1.** *Let $\{K_i\}_{i=1}^m$ be a family of sets in $\mathbb{R}^n$ and let $\theta \in (0,1]$. We define their $\theta$-convolution as the set*

$$K_1 +_\theta \cdots +_\theta K_m = \{x \in \mathbb{R}^n : \chi_{K_1} * \cdots * \chi_{K_m}(x) \geq \theta M(K_1, \dots, K_m)\}$$

*where $M(K_1,\dots,K_m) = \max_{x\in\mathbb{R}^n} \chi_{K_1} * \cdots * \chi_{K_m}(x)$. For $\theta = 0$ we denote by $K_1 +_0 \cdots +_0 K_m$ the support of the function $\chi_{K_1} * \cdots * \chi_{K_m}$, the usual Minkowski sum $K_1 + \cdots + K_m$.*

The commutative and associative properties of the convolution imply trivially that

- $(i)$ $\chi_{K_1} * \cdots * \chi_{K_m} = \chi_{K_{\sigma(1)}} * \cdots * \chi_{K_{\sigma(m)}}$ for any $\sigma$ permutation of $\{1, ..., m\}$.
- $(ii)$ $\chi_{z+K_1} * \cdots * \chi_{K_m}(x) = \chi_{K_1} * \cdots * \chi_{K_m}(x-z)$
- $(iii)$ $\chi_{TK_1} * \cdots * \chi_{TK_m}(x) = |\det T|^{m-1} \chi_{K_1} * \cdots * \chi_{K_m}(T^{-1}x)$ for any $T \in GL_n(\mathbb{R})$.

Consequently we have the following result, analogous to Proposition 2.1.

**Proposition 5.2.** *Let $\{K_i\}_{i=1}^m$ be compact sets in $\mathbb{R}^n$, $\lambda \in \mathbb{R}, \theta \in [0,1]$, $x \in \mathbb{R}^n$ and $T \in GL_n(\mathbb{R})$. Then:*

- $(i)$ $(\lambda K_1) +_\theta \cdots +_\theta (\lambda K_m) = \lambda(K_1 +_\theta \cdots +_\theta K_m)$
- $(ii)$ $K_1 +_\theta \cdots +_\theta K_m = K_{\sigma(1)} +_\theta \cdots +_\theta K_{\sigma(m)}$ *for any $\sigma$ permutation of $\{1, ..., m\}$.*
- $(iii)$ $(x + K_1) +_\theta K_2 +_\theta \cdots +_\theta K_m = x + (K_1 +_\theta \cdots +_\theta K_m)$
- $(iv)$ $TK_1 +_\theta \cdots +_\theta TK_m = T(K_1 +_\theta \cdots +_\theta K_m)$

The convexity is transmitted to the $\theta$-convolution of $m$ convex bodies.

**Proposition 5.3.** *Let $\{K_i\}_{i=1}^m$ be convex bodies in $\mathbb{R}^n$. Then $K_1 +_\theta \cdots +_\theta K_m$ is a convex body.*

*Proof.* The characteristic function of each convex body $K_i$ is log-concave. The convolution of log-concave functions is log-concave, and the level sets of log-concave functions are convex. □

In order to generalize Corollary 2.4 to more than two convex bodies, the following problem arises: Even though for two convex bodies, $\chi_{K_1} * \chi_{K_2}$ is $\frac{1}{n}$-concave in its support, we don't know if the convolution of more than two characteristic functions is $s$-concave for some s. However, its log-concavity property allows us to prove the following:

**Proposition 5.4.** *Let $\{K_i\}_{i=1}^m$ be convex bodies in $\mathbb{R}^n$ such that $M(K_1,\dots,K_m) = \chi_{K_1} * \cdots * \chi_{K_m}(0)$. Then, for every $0 \le \theta \le \theta_0 \le 1$, we have*

$$\frac{K_1 +_\theta \cdots +_\theta K_m}{\log\frac{1}{\theta}} \subseteq \frac{K_1 +_{\theta_0} \cdots +_{\theta_0} K_m}{\log\frac{1}{\theta_0}}.$$

*Proof.* Let $x \in K_1 +_\theta \cdots +_\theta K_m$ and $\alpha \le 1$. Then, from the log-concavity of the convolution we have

$$\begin{aligned}\chi_{K_1} * \cdots * \chi_{K_m}(\alpha x) &\ge \chi_{K_1} * \cdots * \chi_{K_m}(x)^\alpha \chi_{K_1} * \cdots * \chi_{K_m}(0)^{1-\alpha}\\ &\ge \theta^\alpha M(K_1,\dots,K_m).\end{aligned}$$

Hence, taking $\alpha$ so that $\theta^\alpha = \theta_0$ we get

$$\frac{\log\frac{1}{\theta_0}}{\log\frac{1}{\theta}}\,(K_1 +_\theta \cdots +_\theta K_m) \subseteq K_1 +_{\theta_0} \cdots +_{\theta_0} K_m$$

which completes the proof. □

This result provides no information when $\theta \to 0$. However, an analogous proof to that of Theorem 3.4 gives the following

**Theorem 5.5.** *Let $\{K_i\}_{i=1}^m$ be convex bodies in $\mathbb{R}^n$ such that $M(K_1,\dots,K_m) = \chi_{K_1} * \cdots * \chi_{K_m}(0)$. Then for any $\theta \in [0,1]$*

$$(1-\theta^{\frac{1}{(m-1)n}})\,(K_1+\cdots+K_m) \subset K_1 +_\theta \cdots +_\theta K_m.$$

*Proof.* Let $x = \sum_{i=1}^m a_i$ with $a_i \in K_i$. From the convexity of $K_i$,

$$(1-\|a_i\|_{K_i})K_i + a_i \subseteq K_i$$

for all $i$. Hence $\chi_{K_i} \ge \chi_{a_i+(1-\|a_i\|_{K_i})K_i}$ for all $i$ and then, for every $x \in \mathbb{R}^n$,

$$\chi_{K_1} * \cdots * \chi_{K_m}(x) \ge \chi_{a_1+(1-\|a_1\|_{K_1})K_1} * \cdots * \chi_{a_m+(1-\|a_m\|_{K_m})K_m}(x).$$

Since $x = \sum_{i=1}^m a_i$ this previous quantity equals

$$\chi_{(1-\|a_1\|_{K_1})K_1} * \cdots * \chi_{(1-\|a_m\|_{K_m})K_m}(0)$$

which is greater than or equal to

$$\chi_{\min(1-\|a_i\|_{K_i})K_1} * \cdots * \chi_{\min(1-\|a_i\|_{K_i})K_m}(0).$$

This expression equals

$$\min(1-\|a_i\|_{K_i})^{(m-1)n} M(K_1,\dots,K_m).$$

If $a_i \in (1-\theta^{\frac{1}{(m-1)n}})K_i$ the minimum is greater than or equal to $\theta$ and this concludes the proof. $\square$

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